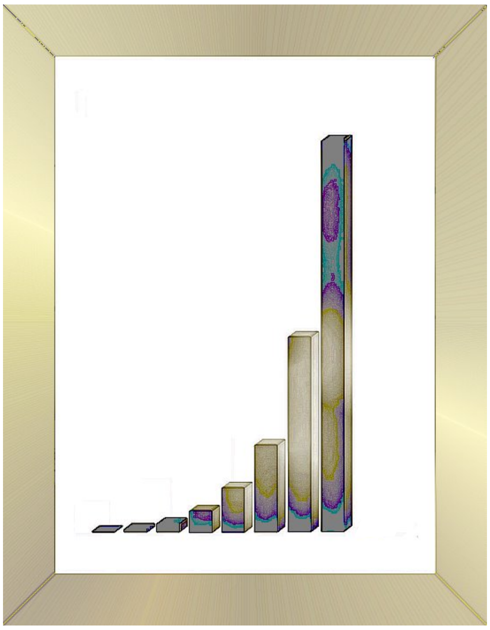
# Exponential Mathematics - I
# "The World's Most Important Arithmetic"

Our school and K-12 curricula help develop student competence in an "arithmetic" mathematics that comprises a simple kind of "grocery-store" arithmetic that is applicable to our daily lives. A far more powerful type of mathematics exists, however, that has extraordinary importance in the modern world. Even in highly literate societies, however, expertise in this *exponential*, logarithmic, or non-linear mathematics receives far too little emphasis.



**Exponential** mathematics is necessary to properly understand human population growth between 8,000 B.C. and 2,000 A.D. This same mathematics also applies to phenomena such as nuclear detonations, algal blooms, proliferating cancer cells, monetary inflation, the pH scale in chemistry, chain reaction explosions, radioactive decay rates, the Richter earthquake scale, compounding interest rates, dinoflagellate red-tides, and human population growth over a span of ten millennia.

Dr. Albert Bartlett, Professor of Physics and Astrophysics Emeritus at the University of Colorado, has called the mathematics of the exponential function "*the world's most important arithmetic.*" In the pages that follow, we will contrast the "grocery-store" mathematics of our early schooling with the powerful and deceptive characteristics of exponential mathematics.

### Linear Mathematics

The mathematics of our elementary schooling centers on operations such as addition, subtraction, multiplication, and division – those simple and non-sophisticated types of math that we might categorize as elementary-school arithmetic. It turns out, however, that such simple math can be, in its way, highly dangerous.

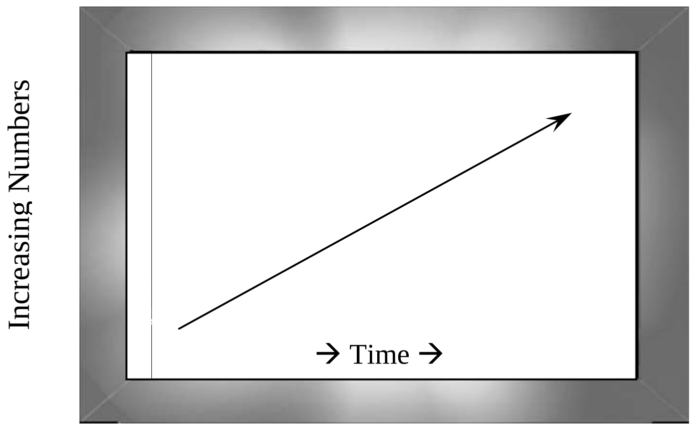
How can "2 + 2" or "4 x 12" possibly be dangerous? Such understandings, after all, make up the mathematics of our daily lives. Aren't such calculations a "fundamental" that all of us should master? While such mathematics clearly applies to our role as consumers, *it is dangerous precisely because we teach it so thoroughly and so well*.

From childhood on, we are taught such "grocery-store" arithmetic so completely and so exclusively, that our minds are literally wired to "interpret the world" using the mathematics of our daily lives. If we try, however, to apply this elementary-school expertise to numbers that behave exponentially, we arrive at <u>dangerously incorrect answers</u>. As a result, one of our dangers is this: With each passing year, and each new homework set, we are wiring our students' brains with the same mathematics that prepared their great grandparents for life in the 1930s and a different time in history.

Because we place so much emphasis on the skills of addition, subtraction, multiplication, and division and other traditional mathematics of the past, *we leave today's students ill-equipped to deal with* (or to even perceive)

the critical REAL-WORLD mathematics of the twenty-first century -
the mathematics of the EXPONENTIAL FUNCTION.

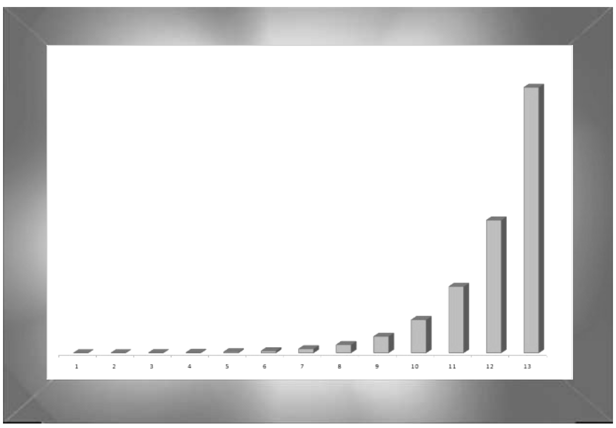
Some number sequences grow in a traditional way, by repeatedly adding a like amount to a growing variable, and generate a linear graph like the one below.




Linear graphs are characteristic of number sequences that increase by repeated ***additions*** of like amounts, such as: 3… 6...9…12…15…18…21… etc. In this example, the numbers are growing by repeated additions of three. When numbers grow (or decline) in this way, we say that they are growing *arithmetically* and when we graph such a sequence over time, the graph they produce, like the one above, is **linear** – that is, a straight line.

**Exponential Mathematics**

Numbers that increase "exponentially" grow by repeated *multiplications* by like amounts. An example of an exponential sequence is 1...2...4...8...16...32...64.. etc. In this example, notice that each number has been multiplied by two. When we graph an exponential progression, we obtain a **J-curve** like the one shown below.



Notice that an exponential graph is shaped a little like the letter "J" so that initially, *it rises slowly from the x-axis* like an airplane rising from a runway. Toward the end of the data set, however, numbers that grow exponentially turn sharply upward like a rocket. If we graph human population growth between 8000 B.C. and 2000 A.D. (using, for example, from other PDFs in this series), we generate a graph like the one shown below.

The mathematics that produces graphs of this sort is our topic in the four "Exponential" PDFs in this set. just ahead (I, II, III, IV). We will see that exponential number sequences are misleading and deceptive, exceedingly powerful, and frequently dangerous. To properly evaluate today's demographic and environmental issues, we must be able to analyze numbers in an exponential context. We must also appreciate the special properties of exponential progressions in order to place humanity's current trajectories in their proper historical context. Two examples will help us contrast arithmetic versus exponential patterns of growth.

### 1 - An "Arithmetic" Progression

*Linear* mathematics is illustrated by the following example: Suppose one is offered a contract with a salary for thirty days of work over a one-month period. Furthermore, this pay is to *increase* each day as follows: On day one we receive $1000 and on day two we receive $2000. (Notice that we have just earned three thousand dollars over the first two days.) On day three our pay is $3000; on day four we earn $4000, etc. Since this salary increases by a pattern of repeated additions of like amounts ($1000) we say that it is growing **ARITHMETICALLY**. In this case, the salary grows steadily and predictably larger at a straight-forward rate of an extra one thousand dollars per day. Thus, the pay for day thirty alone will be $30,000 while the total for all thirty days will amount to $465,000.

### 2 - An Exponential Alternative

Suppose that a mathematician suggests asking for a more humble salary that grows larger **EXPONENTIALLY** each day. In this case, one's initial salary will be painfully small with a total day one salary of just one cent. However, for one's work on day two, this increases to two cents. On day three the pay for the entire day is four cents. (Notice that the total for the first three days is only seven cents.) Then, on day four, the pay climbs to eight cents and jumps again to sixteen cents on day five. Because the numbers in this progression are growing by repeated multiplications by like amounts, we say that the pattern of growth is *exponential.*

### Contrasting the Options

Let us now contrast the two options. We want to know which option will be more rewarding. How much will each pattern generate in the effective month? The first progression is exceptionally generous at the outset, but will it remain most generous by month's end? Since some months are thirty-one days long, how much difference will a day make?

During the first seven days, the exponential option generates a total of just $1.27 – or about two cents an hour. During the same seven days, however, the arithmetic option generates $28,000. Thus, by the end of the first week, the net result is: *arithmetic growth* = $28,000; *exponential growth* = $1.27.

This disparity continues during week two. By the end of the first fourteen days, the arithmetic option amounts to $105,000 while the exponential salary generates a total income of just $163.83 – (averaging a little less than $1.50 an hour). Thus, nearly halfway through the month, the tally stands at: *arithmetic growth* = $105,000; *exponential growth* = $163.83.

At this point, the exponential progression seems to be a poor choice. Anyone who chose a salary that grows exponentially, has, so far, very little to show for it. During the remainder of the month, however, the results will be quite different.

## The Remainder of the Month

The reader is invited to use a calculator to compare the two salaries over the remaining days of the month. Knowing that most of us are busy, however, we have calculated some of the results as follows: On day twenty-five, the exponential option generates more than $167,000 as the single-day salary due on that day alone. Then:

| | |
|---|---|
| Day 26 | 335,544.32 |
| Day 27 | 671,088.64 |
| Day 28 | 1,342,177.28 |
| Day 29 | 2,684,354.56 |
| Day 30 | 5,368,709.12 |

Initially the *arithmetic* alternative, with a thirty-day total of $465,000 looked quite attractive. Now, however, it becomes clear that the exponential option (that began with such tiny amounts) *will reward one with more than ten million dollars* **($10,737,418.23)** during the same period. When was most of this exponential salary generated? When did most of the growth take place?

The values peaked explosively in
the **closing stages** of the sequence.

In the mid-twentieth century, there was a popular blues piece entitled "*What difference does a day make*?" The song was about a love affair, but its title presents an interesting question in the context of a salary that grows exponentially. If a set of numbers grows exponentially, what difference does a day make? If we choose a month that is thirty-one days long, the extra day *will double our salary again* to $10,737,418.24. (This will also cause the thirty-one-day **total** to also *double*, growing to more than **$21,000,000** dollars.)

## Summarizing Exponential Patterns

Let us summarize several key observations: First, notice that although the numbers making up the **linear** sequence seem attractively large at the outset and present themselves in obvious terms, the initial numbers of the EXPONENTIAL data set are so small that they seem to be harmless or unimportant.

This means that numbers that grow exponentially can be **deceptive** and seriously misleading. Because the original numbers can be quite small, *they invite us to suppose that they* are innocuous and *require little attention*. We have just seen, however, a number set begin with one cent and then turn itself into more than twenty-one million dollars in just thirty-one days. Numbers that behave in this way can be exceedingly dangerous because first they lure us into complacency, *and then they hammer us*.

We should also notice that the exponential salary is still exceptionally modest even after two full weeks. This fact underscores a second crucial characteristic: Most of the growth in an exponential progression occurs toward the end of the progression. As the sequence proceeds, its numbers grow explosively larger, piling up astronomically in the closing stages of the sequence.

Finally, we see that exponential sequences are exceedingly powerful. We just saw one cent transformed into more than $21,000,000 in just thirty-one days, with most of these dollars piling up in the closing stages of the progression. (And the sample exercise also showed a startling transformation from modest values to utter calamity in just two weeks.) This illustrates that exponential growth can turn tiny numbers into exceedingly large numbers in an unexpectedly short time.

Because exponential sequences are both powerful and deceptive, they can also be extremely dangerous. Thus, three key adjectives serve as our summary so far: Exponential numbers behave in a way that is, simultaneously, (a) **powerful**, (b) **deceptive**, and (c) **dangerous**.

### The Hiroshima Weapon

Events inside explosions and nuclear detonations are not linear / arithmetic events. Instead, they are EXPONENTIAL events that, when graphed, produce a J-curve like thoswe depicted earlier. Thus the nuclear fission events inside the atomic weapon that was dropped on Hiroshima, Japan at the close of World War II followed an exponential pattern, releasing enough energy to suddenly destroy the city and kill tens of thousands of people. In one sense, the energy released by the atomic fission destroyed the city.

In another sense, however, it was the *exponential*
nature of the fission events that destroyed the city.

These characteristics should alarm us because *human population growth* over the millennia between 8000 B.C. and 2000 A.D. has been exponential (or even, as Joel Cohen points out, "hyperexponential"). In our chapter three (and an earlier excerpt) we sketched mankind's demographic journey through human history. When we graph the data from that chapter, beginning with earth's human population in 8000 B.C. and ending in 2000, we obtain a classical J-curve. Campbell, Reece, and Mitchell (1999) described it this way:

> "The exponential growth model... describes the population explosion in humans. Ours is a singular case; it is unlikely that any other population of large animals has ever sustained so much growth for so long."

We should find this disquieting because, in the strictly mathematical sense of the term, we may be in the closing stages of a detonation that is not unlike the detonation that flattened Hiroshima, Japan in World War II. As this is written, most of the industrialized world appears to have retreated from this exponential track, but in more than fifty nations, especially in the Middle East, parts of Asia, and parts of Africa, populations are still rocketing upward.

### A Collision Course

In recent decades, top scientists from around the world have written papers and issued formal warnings about human population growth and our impacts on our planet. For example, “catastrophic events share characteristic nonlinear behaviors [that] result in surprises that cannot be easily predicted” (Peters, et al., 2004). And a decade earlier, more than 1500 top scientists, including 99 recipients of the Nobel prize issued an "*Urgent warning to humanity*" which we excerpt as follows:

"Human beings and the natural world are on a collision course. Human activities inflict harsh and often irreversible damage on the environment and on critical resources. If not checked, many of our current practices put at serious risk the future that we wish for human society and the plant and animal kingdoms, and may so alter the living world that it will be unable to sustain life in the manner that we know. Fundamental changes are urgent if we are to avoid the collision our present course will bring about.

The signatories then warn that "The earth is finite. Its ability to absorb wastes and destructive effluent is finite. Its ability to provide food and energy is finite. Its ability to provide for growing numbers of people is finite. And we are fast approaching many of the earth's limits."

Pressures resulting from unrestrained population growth put demands on the natural world that can overwhelm any efforts to achieve a sustainable future. If we are to halt the destruction of our environment, we must accept limits to that growth.

No more than one or a few decades remain before the chance to avert the threats we now confront will be lost and the prospects for humanity immeasurably diminished."

We, the undersigned, senior members of the world's scientific community, hereby warn all humanity of what lies ahead. A great change in our stewardship of the earth and the life on it is required if vast human misery is to be avoided and our global home on this planet is not to be irretrievably mutilated" (Kendall, et al., 1992).

* In a similar way, scholars repeatedly tried to warn government officials and residents of New Orleans about category five hurricanes and the insufficiency of the city's levee system for more than a decade before hurricane Katrina struck in 2005 – underscoring, perhaps, the fact that governments and institutions have something less than a 100% track record in their responses to warnings.

In the years since the Kendall, et al. warning, we have added more than one BILLION additional people to our planet and according to current U.N. projections, our seventh, eighth, and ninth billions are on-track to arrive between now and mid-century.

A continuation of today's demographic tidal wave *may constitute the greatest single risk that our species has ever undertaken.*

Excerpted from
What Every Citizen Should Know About Our Planet
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Expanded implications of this excerpt are also addressed in additional PDFs in this collection:

- Razor-Thin Films: Earth's Atmosphere and Seas
- Numerics, Demographics, and a Billion Homework Questions
- Conservation planning - Why Brazil's 10% is Not Enough
- Eight Assumptions that Invite Calamity
- Climate - No Other Animals Do This
- Critique of Beyond Six Billion
- Delayed feedbacks, Limits, and Overshoot
- Thresholds, Tipping points, and Unintended consequencs
- Problematic Aspects of Geoengineering
- Carrying Capacity and Limiting Factors
- Humanity's Demographic Journey
- Ecosystem services and Ecological release
- J-curves and Exponential progressions
- One hundred key Biospheric understandings

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